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Lorentz's theorem on the Stokes equation

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Abstract. A simple derivation of the Lorentz theorem is presented which gives the perturbation pressure and velocity due to the presence of a plane wall introduced into an unlimited viscous fluid of given pressure and velocity obeying the Stokes equation.
An extension to the case of a spherical boundary is made in the same manner, leading to the case of a plane boundary as a limit of large radius. The sphere theorem is revised and applied to three elementary solutions by Lamb.

1. Introduction

In 1896 Lorentz [1] developed a theorem yielding a mirror image of the pressure and velocity of the Stokes flow due to a plane wall in a unlimited viscous fluid. In a previous review [2] this theorem has been shown to be derived by the use of the general solution in terms of three harmonics proposed by Imai [3].

As to the case of a spherical boundary, the present author [4] presented a sphere theorem for the perturbed stream function of the axisymmetric flow due to a sphere introduced into an unlimited viscous flow in axisymmetric motion, obeying the Stokes equation. This theorem has been presented by Collins in a more compact form [5].

Recently, Paraniappan et al [6,7] have extended this theorem to a general non-axisymmetric flow represented by a biharmonic function and a harmonic function, on the basis of an inversion theorem for the polyharmonics due to Chwang[8]. This formalism has been applied by their group to several internal flows [9] and has ben extended to the case of a spherical interface [10].

In this paper will be presented a simple procedure giving the reflected pressure and velocity due to a plane or spherical boundary [11, 12] directly from the original flow, without recourse to auxiliary functions, in a similar way to the original Lorentz formula.

The sphere theorem is given in an alternative form and is applied to three types of elementary solutions given by Lamb [13].

2. Plane wall

2.1. Preliminaries

Let us derive the image field due to the presence of a plane wall $z = 0$ in the Cartesian system (x, y, z) with the unit vector **e** normal to the wall. For this purpose, it is convenient to define the reflection of any function $f(x, y, z)$ by $f^*(x, y; z) = f(x, y, -z)$ and start from the following lemmas for any harmonic function $H(x, y, z)$ and biharmonic function $B(x, y, z)$.

LEMMA 1. *$\tilde{H} = - H^*$ is harmonic:*

$$\Delta\tilde{H} = 0 \tag{2.1}$$

and satisfies $\tilde{H} + H = 0$ at $z = 0$.

LEMMA 2. *Let $\tilde{B} = -[B - 2zB' + z^2\Delta B]^*$,*
where the prime $'$ denotes the z derivative $\mathbf{e} \cdot$ grad.
Then $\tilde{B}$ is biharmonic

$$\Delta^2\tilde{B} = 0 \tag{2.2}$$

and satisfies $\tilde{B} + B = 0$ as well as $\tilde{B}' + B' = 0$ at $z = 0$.

LEMMA 3.

$$\Delta\tilde{B} = [4B'' - 3\Delta B - 2z\Delta B']^* \tag{2.3}$$

2.2. LORENTZ'S THEOREM

Let us start from the velocity $\mathbf{u}$ and the pressure p of the viscous flow satisfying the continuity and the Stokes equation:

$$\operatorname{div}\mathbf{u} = 0 \tag{2.4}$$

and

$$\Delta\mathbf{u} = \operatorname{grad} p/\mu \tag{2.5}$$

or

$$\operatorname{rot}\boldsymbol{\omega} = -\operatorname{grad} p/\mu, \tag{2.6}$$

where $\boldsymbol{\omega}$ is the vorticity

$$\boldsymbol{\omega} = \operatorname{rot}\mathbf{u}. \tag{2.7}$$

Then it is easily seen that
i) $p, p' = \mathbf{e} \cdot \operatorname{grad} p$, $\mathbf{e} \times \operatorname{grad} p$, $\boldsymbol{\omega}$ and $\omega_z = \mathbf{e} \cdot \boldsymbol{\omega}$ are all harmonic.
ii) $\mathbf{u}$, $w = \mathbf{u} \cdot \mathbf{e}$, $\mathbf{e} \times \mathbf{u}$ and $z\boldsymbol{\omega}$ are all biharmonic, satisfying (2.2) as well as

$$\Delta(\mathbf{e} \cdot \mathbf{u}) = p'/\mu, \tag{2.8}$$

and

$$\Delta(z\boldsymbol{\omega}) = 2\boldsymbol{\omega}'. \tag{2.9}$$

It should be noted that these properties are valid also for the perturbed quantities denoted by a tilde $\sim$.

Applying Lemma 2 to the biharmonic function $w = \mathbf{e} \cdot \mathbf{u}$ satisfying $\tilde{w} + w = 0$ and $(\tilde{w} + w)' = 0$ at $z = 0$, we have

$$\tilde{w} = -\mathbf{e} \cdot \mathbf{q}[\mathbf{u}, p]^*, \tag{2.10}$$

where

$$\mathbf{q}[\mathbf{u},p] = \mathbf{u} - 2z\mathbf{u}' + z^2 \mathrm{grad}\, p/\mu, \tag{2.11}$$

and we have made use of (2.8).

The application of Lemma 3 to (2.10) yields

$$\tilde{p}' = \mu\Delta\tilde{w} = [4\mu w'' - 3p' - 2zp'']^*. \tag{2.12}$$

Integrating (2.12) with respect to $z = -z^*$, we have

$$\tilde{p} = [p + 2zp' - 4\mu w']^* = [(2zp - 4\mu w)' - p]^* \tag{2.13}$$

Making use of the relation

$$\mathbf{e} \times \mathrm{grad}\, p/\mu = (\mathbf{e}\cdot \mathrm{grad}\,)\boldsymbol{\omega} - \mathrm{grad}\,(\mathbf{e}\cdot\boldsymbol{\omega}) \tag{2.14}$$

derived from (2.6), we have for the perturbed vorticity $\boldsymbol{\omega}$

$$\tilde{\boldsymbol{\omega}} = \int [\mathbf{e} \times \mathrm{grad}\,\tilde{p}/\mu + \mathrm{grad}\,(\mathbf{e}\cdot\tilde{\boldsymbol{\omega}})]\mathrm{d}\, z, \tag{2.15}$$

where the z-component $\mathbf{e}\cdot\tilde{\boldsymbol{\omega}}$ in the integrand is found to be

$$\mathbf{e}\cdot\tilde{\boldsymbol{\omega}} = -\mathbf{e}\cdot\boldsymbol{\omega}^* \tag{2.16}$$

from Lemma 1 applied to the harmonic function satisfying

$$\mathbf{e}\cdot\tilde{\boldsymbol{\omega}} = -\mathbf{e}\cdot\boldsymbol{\omega} \qquad \text{at } z = 0. \tag{2.17}$$

Introducing (2.13) and (2.16) in (2.15), we obtain the tangential component

$$\boldsymbol{\omega}_t = \boldsymbol{\omega} - (\mathbf{e}\cdot\boldsymbol{\omega})\mathbf{e}$$

$$\tilde{\boldsymbol{\omega}}_t = [\boldsymbol{\omega}_t + \mathbf{e}\times \mathrm{grad}\,(4w - 2zp/\mu)]^* \tag{2.18}$$

where we have made use of (2.13).

The tangential velocity can be obtained by applying Lemma 2 to the biharmonic function $\mathbf{U}$:

$$\mathbf{U} = \mathbf{e}\times\mathbf{u} - z\boldsymbol{\omega} \tag{2.19}$$

which satisfies the boundary condition

$$\tilde{\mathbf{U}} + \mathbf{U} = 0 \quad \text{and} \quad (\tilde{\mathbf{U}} + \mathbf{U})' = 0 \quad \text{at} \quad z = 0. \tag{2.20}$$

Making use of (2.8), (2.9) and (2.11), we have

$$\tilde{\mathbf{U}} = -[\mathbf{e}\times\mathbf{q}(\mathbf{u},p) + z\boldsymbol{\omega}]^* \tag{2.21}$$

which yields by noting (2.19)

$$\begin{aligned} \mathbf{e}\times\tilde{\mathbf{u}} &= \mathbf{U} + z\tilde{\boldsymbol{\omega}} \\ &= -\mathbf{e}\times[\mathbf{u} - 2z\mathbf{u}' + z^2\mathrm{grad}\,p/\mu]^* + z(\tilde{\boldsymbol{\omega}} + \tilde{\boldsymbol{\omega}}^*), \end{aligned} \tag{2.22}$$

where $\tilde{\boldsymbol{\omega}}$ is given by (2.16) and (2.18).

Combining (2.10) and (2.22), we have

$$\tilde{\mathbf{u}} = -[\mathbf{u} - 2z\mathbf{u}' + z^2 \text{grad}\, p/\mu]^* - z\mathbf{e} \times (\tilde{\omega} + \tilde{\omega}^*), \qquad (2.23)$$

or

$$\tilde{w} = -[w - 2zw' + z^2 p'/\mu]^*, \qquad (2.24)$$

and

$$\mathbf{u}_t = -[\mathbf{u}_t + 2z\, \text{grad}_t w - z^2\, \text{grad}_t p/\mu]^*, \qquad (2.25)$$

where we have made use of (2.18) and (2.14).

The expressons (2.13), (2.24) and (2.25) for $\tilde{p}$, $\tilde{w}$ and $\tilde{\mathbf{u}}_t$ are essentially the formulae given by Lorentz. [1]

3. Spherical boundary

3.1. Preliminaries

It is convenient to rewrite Kelvin's theorem for the harmonics $H(\mathbf{x})$ and Chwang's theorem for the bihamonics $B(\mathbf{x})$ in the following form:

THEOREM 1 (Kelvin's exterior theorem). *Let $H(\mathbf{x})$ be the harmonic function of $\mathbf{x} = r\mathbf{e}$ and be regular in the domain $r < a$. Then*

$$\tilde{H}(\mathbf{x}) = -aH(\mathbf{x}^*)/r \qquad (3.1)$$

is regular harmonic in $r > a$ and satisfies the boundary condition

$$\tilde{H} + H = 0 \quad \text{at } r = a, \qquad (3.2)$$

where r is the radius from the center and $\mathbf{e}$ is the unit radiul vector.

*Here and hereafter the asterisk * denotes the inversion*

$$[f(\mathbf{x})]^* = f(\mathbf{x}^*)$$

with

$$\mathbf{x}^* = a^2\mathbf{x}/r^2 = r^*\mathbf{e}$$

and

$$r^* = a^2/r. \qquad (3.3)$$

THEOREM 2 (Chwang's exterior theorem). *Let $B(\mathbf{x})$ be the biharmonic function satisfying*

$$\Delta^2 B = 0 \qquad (3.4)$$

and regular in the doman $r < a$. Then $\tilde{B}$ given by

$$\tilde{B} = -[(r/a + a/r)B/2 + a(1 - r^2/a^2)B' + a^3(1 - r^2/a^2)^2 \Delta B/(4r)]^* \qquad (3.5)$$

is regular biharmonic in $r > a$ and satisfies the following boundary conditions on the sphere $r = a$:

$$\tilde{B} + B = 0 \quad \text{and } (\tilde{B} + B)' = 0 \quad \text{at } r = a \qquad (3.6)$$

where the prime ′ denotes the derivative $\partial/\partial r = (1/r)(\mathbf{x} \cdot \text{grad}) = \mathbf{e} \cdot \text{grad}$.

Theorem 2 is a generalization of the sphere theorem for the axisymmetric stream function in the Stokes flow derived by the author [4] and Collins [5].

We may note the following formula for $\Delta\tilde{B}$:

$$\Delta\tilde{B} = -[(rB - 4r^2B' - 4r^3B'')/a^3 + (r/a + 5r^3/a^3)\Delta B/2 \\ -a(r^2/a^2 - r^4/a^4)(\Delta B)']^*, \tag{3.7}$$

which is easily derived from (3.4) and (3.3) by use of equalities

$$\Delta(f^*/r) = a^4(\Delta f)^*/r^5 \tag{3.8}$$

and

$$\Delta(rf') = \Delta(\mathbf{x}\cdot \text{grad}\,)f] = (\mathbf{x}\cdot\text{grad}\, + 2)\Delta f. \tag{3.9}$$

INTERIOR THEOREMS 1′ AND 2′

In the interior problem where we interchange $r > a$ and $r < a$ in the above Theorems we have to restrict H and B to be respectively $0(1/r)$ and $0(r)$ as $r \to \infty$.

3.2. SPHERE THEOREM

If we replace $z\,\mathbf{e}$ in the section 2 by $\mathbf{x}$ the procedure proceeds analogously to the case of a plane boundary. We have

i) $p, rp' = \mathbf{x}\cdot\text{grad}\,p, \mathbf{x}\times\text{grad}\,p$ and $\mathbf{x}\cdot\boldsymbol{\omega} = r\omega_r$ are all harmonic.

ii) $\mathbf{u}, ru_r = \mathbf{x}\cdot\mathbf{u}, \mathbf{x}\times\mathbf{u}$ and $r^2\boldsymbol{\omega}$ are all biharmonic, satisfying (3.4) as well as

$$\Delta(\mathbf{x}\cdot\mathbf{u}) = rp'/\mu, \tag{3.10}$$

$$\Delta(\mathbf{x}\times\mathbf{u}) = 2\boldsymbol{\omega} + \mathbf{x}\times\text{grad}\,p/\mu. \tag{3.11}$$

and

$$\Delta(r^2\boldsymbol{\omega}) = 4r\boldsymbol{\omega}' + 6\boldsymbol{\omega}. \tag{3.12}$$

It should be noted that these properties are valid also for the perturbed quantities denoted by a tilde $\sim$.

Applying the theorem 2 to the biharmonic function $\mathbf{x}\cdot\mathbf{u}$ satisfying $\mathbf{x}\cdot(\tilde{\mathbf{u}}+\mathbf{u}) = 0$ and $\mathbf{x}\cdot(\tilde{\mathbf{u}}+\mathbf{u})' = 0$ at $r = a$, we have

$$\tilde{u}_r = \mathbf{x}\cdot\tilde{\mathbf{u}} = \{\mathbf{x}\cdot\mathbf{q}[\mathbf{u},p]\}^*, \tag{3.13}$$

where

$$r\mathbf{q}[\mathbf{u},p] = a\{(3 - r^2/a^2)\mathbf{u}/2 + (1 - r^2/a^2)r\mathbf{u}' + a^2(1 - r^2/a^2)^2\text{grad}\,p/(4\mu)\} \tag{3.14}$$

and we have made use of (3.10).

The application of (3.10) and (3.7) yields

$$\tilde{p}' = \mu\Delta(\mathbf{x}\cdot\tilde{\mathbf{u}})/r = \{\mu a(4r^2\mathbf{u}_r'' + 12ru_r' + 3u_r) + a^3r(1 - r^2/a^2)p'' \\ +a^3(1 - 7r^2/a^2)p'/2\}^*/r^3. \tag{3.15}$$

The perturbed pressure $\tilde{p}$ is obtained by the integration of (3.15) with respect to $r = a^2/r^*$.

$$\tilde{p} = -\{(4r^3 u_r' + 3\int r u_r, \mathrm{d}r)/a^3 + a(r^2/a^2 - r^4)p' - I/(2a)\}^*$$

with

$$I = \int (3 - r^2/a^2) r p' \,\mathrm{d}r = (3 - r^2/a^2) r p - 3\int (1 - r^2/a^2) p \,\mathrm{d}r, \tag{3.16}$$

where we assume that the integrals are convergent and take the gauge pressure corresponding to the lower limit (r =0 or infinity according as outer or inner problem) to be zero.

Making use of the relation

$$\mathbf{x} \times \mathrm{grad}\, p/\mu = (r\boldsymbol{\omega})' - \mathrm{grad}\,(\mathbf{x}\cdot\boldsymbol{\omega}) \tag{3.17}$$

derived from (2.6), we have for the perturbed vorticity:

$$r\tilde{\boldsymbol{\omega}} = \int [\mathbf{x}\mathrm{grad}\,\tilde{p}/\mu - \mathrm{grad}\, a(\mathbf{x}\cdot\boldsymbol{\omega})^*/r]\,\mathrm{d}r, \tag{3.18}$$

since $\mathbf{x}\cdot\boldsymbol{\omega} = r\omega_r$ is harmonic and satisfy the condition (3.2) of theorem 1 on the sphere $r = a$. We have

$$r\tilde{\omega}_r = \mathbf{x}\cdot\tilde{\boldsymbol{\omega}} = -a(\mathbf{x}\cdot\boldsymbol{\omega})^*/r, \tag{3.19}$$

from theorem 1.

The tangential velocity $\mathbf{u}_t$ can be obtained by applying theorem 2 to the biharmonic function $\mathbf{V}$

$$\mathbf{V} = \mathbf{x}\times\mathbf{u} - (r^2 - a^2)\boldsymbol{\omega}/2 \tag{3.20}$$

satisfying the boundary conditions of the theorem. We get

$$\tilde{\mathbf{V}} = -\{\mathbf{x}\times\mathbf{q}[\mathbf{u},p] - a(r/a - r^3/a^3)\boldsymbol{\omega}/2\}^*, \tag{3.21}$$

where $\mathbf{q}[\mathbf{u},p]$ is given by (3.14) and we have made use of (3.11) and (3.12).

Combining the expression for $\mathbf{x}\times\mathbf{u}$ obtained from (3.20), (3.21) and (3.13), we have

$$\begin{aligned}\tilde{\mathbf{u}} &= -\{(3r/a - r^3/a^3)\mathbf{u}/2 + (r^2/a^2 - r^4a^4)a\mathbf{u}' + r(a^2 - r^2)^2\mathrm{grad}\,p/(4\mu a^3)\}^* \\ &\quad -[(r/a - r^3/a^3)\{a^3\mathrm{grad}_t \int \tilde{p}^*/(\mu r^2)\mathrm{d}r + \mathbf{e}\times(r\boldsymbol{\omega} + \mathrm{grad}\int \mathbf{x}\cdot\boldsymbol{\omega}\mathrm{d}r)\}]^*/2,\end{aligned} \tag{3.22}$$

where grad_t denotes the tangential derivative grad - $\mathbf{e}$ ($\mathbf{e}$ $\cdot$grad).

Separating the radial and the tangential component we have

$$\tilde{u}_r = \tilde{\mathbf{u}}\cdot\mathbf{e} = -\{(3r/a - r^3/a^3)u_r/2 + (r^2/a^2 - r^4/a^4)au_r' + r(a^2 - r^2)^2 p'/(4\mu a^3)\}^* \tag{3.23}$$

$$\begin{aligned}\tilde{\mathbf{u}}_t &= \tilde{\mathbf{u}} - \tilde{u}_r\mathbf{e} \\ &= -[r\mathbf{u}_t/a - (1 - r^2/a^2)\mathrm{grad}_t[(r^2u_r - 3\int r u_r/2\mathrm{d}r)/a \\ &\quad + a\{r(1 - r^2/a^2)p - 3\int (1 - r^2/a^2)p\,\mathrm{d}r\}/(4\mu)\}]]^*\end{aligned} \tag{3.24}$$

where we have made use of the relation

$$\mathbf{x} \times \boldsymbol{\omega} = -(r\mathbf{u})' + \text{grad}\,(\mathbf{x} \cdot \mathbf{u}) \tag{3.25}$$

as well as (3.17) to eliminate the vorticity and (3,16) to eliminate p, and assume that the constant of integration can be taken to be zero at the lower limit*.

The expression (3.24) is new and corresponds directly to (2.25) of the previous section.

3.3. Limit for large radius

Let us put

$$r = a(1 + z/a + o(z/a)) \tag{3.26}$$

in the several expressions and retain the lowest order in z/a

For the pressure p, it is convenient to adopt the last exprssion of I in (3.16) and neglect the integral in comparison with the first term. We may put $\mathbf{u}$ as w, r - derivative as z - derivative to the lowest approxination, obtaining (2.10) for p.

It is easily seen that the radial velocity (3.13) leads to the expression (2.24) for w. In the same manner, (2.25) is derived from (3.24) by neglecting integral terms of $O(z/a)$ in comparison with other terms.

3.4. Example

As an illustrating example, three components of Lamb's general solution [13] are considered. For the sake of simplicity we may take a and μ to be unity without loss of genarility.

1) External flow $r > 1$.

$i)$ $\mathbf{u} = \text{rot}\,(\mathbf{x}H) = \mathbf{x} \times \text{grad}\,H = -\mathbf{C}$,

where H and $\mathbf{C}$ are solid harmonics of n-th degree (n is a positive integer, since n= 0 is trivial). We have also

$$p = 0 \quad \text{and} \quad u_r = 0.$$

Introducing these into the corresponding expressions in 3.2, we obtain

$$\tilde{p} = 0 \quad \text{and} \quad \tilde{u}_r = 0 \quad \text{so that} \quad \tilde{\mathbf{u}}_t = [r\mathbf{C}]^*,$$

i.e.

$$\tilde{\mathbf{u}} = -\mathbf{u}^*/r.$$

$ii)$ $\mathbf{u} = \text{grad}\,H$,

We have

$$u_r = nH/r, \quad \mathbf{u}_t = \text{grad}_t H = \mathbf{T}/r$$

with

$$\mathbf{T} = r\,\text{grad}_t H$$

as well as

$$p = 0,\ u_r' = n(n-1)H/r^2, \quad \mathbf{u}_t' = (n-1)\mathbf{T}/r^2$$

and

$$\int r u_r \mathrm{d}r = nrH/(n+1)$$

Then (3.16), (3,23) and (3.24) give

$$\tilde{p} = -n(4n^2-1)(rH)^*/(n+1),$$

$$\tilde{u}_r = -n[\{n + 1/2 - (n-1/2)r^2\}H]^*$$

and

$$\begin{aligned}\tilde{\mathbf{u}}_t &= -[\{1 - n(1-r^2) + 3n(1-r^2)/(2n+2)\}\mathbf{T}]^* \\ &= [\{(n-2)(2n+1) - n(2n-1)r^2\}\mathbf{T}]^*/(2n+2).\end{aligned}$$

iii) $p = H$,

$$\mathbf{u} = [(n+3)r^2 \mathrm{grad}\, H - 2n\mathbf{x}H]/[(2n+2)(2n+3)].$$

We have

$$u_r = nrH/(4n+6), \quad \mathbf{u}_t = (n+3)r\mathbf{T}/[(2n+2)(2n+3)],$$

$$\int r u_r = nr^3H/[(n+3)(4n+6)], \quad \int (1-r^2)p\,\mathrm{d}r = [r/(n+1) - r^3/(n+3)]H.$$

Introducing these expressions in (3.19), (3.23) and (3.24), we obtain

$$\tilde{p} = -n(2n-1)(rH)^*/(2n+2)$$

$$\tilde{u}_r = -n[\{2n+3-(2n+1)r^2\}H]^*/(8n+12)$$

and

$$\tilde{\mathbf{u}}_t = [\{(n-2) - n(2n+1)r^2/(2n+3)\}\mathbf{T}]^*/(4n+4).$$

2) Interior flow $r < 1$.

In this case we have only to take solid harmonics of $-(n+1)$ degree or replace H by $(rH)^*$ etc. in the exterior problem

$$i)\ \mathbf{u} = \mathrm{rot}\,(\times H^*/r) = -\mathbf{x} \times \mathrm{grad}\, H^*/r = -\mathbf{C}^*/r,$$

where H and $\mathbf{C}$ are solid harmonics of n th degree and we obtain

$$\tilde{p} = 0, \tilde{u}_r = 0 \quad \text{so that} \quad \mathbf{u} = \mathbf{C} = -\mathbf{u}^*/r.$$

$$ii)\ \mathbf{u} = \mathrm{grad}\,(H^*/r), \quad p = 0.$$

We have

$$\tilde{p} = -(n+1)(2n+1)(2n+3)H/n,$$
$$\tilde{u}_r = -(n+1)\{(n+1/2)r - (n+3/2)/r\}H$$

and

$$\tilde{\mathbf{u}}_t = -\{(n+3)(n+1/2)r - (n+1)(n+3/2)/r\}\mathbf{T}/n$$

$$iii)\, p = (rH)^*,$$

$$\mathbf{u} = -[(n-2)r^2 \mathrm{grad}\,(rH)^* - (2n+2)\mathbf{x}(rH)^*]/[2n(2n-1)]$$

we obtain

$$\tilde{p} = (n+1)(2n+3)H/(2n)$$
$$\tilde{u}_r = (n+1)\{r^2 - (2n+1)/(2n-1)\}H/(4r)$$

and

$$\tilde{\mathbf{u}}_t = \{(n+3)r^2 - (n+1)(2n+1)/(2n-1)\}\mathbf{T}/(4nr)$$

It is seen that the case $n = 1$ in *iii*) (stokeslet) is formally valid though the convergence condition Ci is violated. This fact suggests to apply our formula to the general inner field with stokeslet behavior S at infinity after subtraction of S and complementing its perturbation $\tilde{S}$ later.

4. Summary

A simple derivation of the Lorentz theorem is presented which gives the perturbation pressure $\tilde{p}$ and velocity $\tilde{\mathbf{u}}$ due to the presence of a plane wall introduced into an unlimited viscous fluid of given pressure p and velocity $\mathbf{u}$, obeying the Stokes equation.

An extension to the case of a spherical boundary is made in the same manner, leading to the case of a plane boundary as a limit of large radius. The theorem is given in the form corresponding to the original formula of Lorentz for the plane boundary.

As an illustrating example three elementary solutions by Lamb [13] were chosen. Application to any case will be easily done, even when the stokeslet behavior violates the convergence condition at infinity in the interior problem [9], if the total *radial* flux is zero.

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Note

* We have only to require the condition Ci in the interior problem: Condition Ci

$$u_r - rp/2 = 0(1/r^3) \quad \text{as} \quad r \to \infty.$$

References

1. H. A. Lorentz, A general theorem concerning the motion of a viscous fluid and a few consequences derived from it. *Zittingsverslag. Akad. Wet. Amsterdam* 5 (1896) 168–175; see also *Abhand. Theor. Phys.* Leipzig: Teubner (1907) pp 23–42.
2. H. Hasimoto and O. Sano, Stokeslet and eddies in creeping flow. *Ann. Rev. Fluid Mech.* 12 (1980) 335–363.
3. I. Imai, Some applications of function theory to fluid dynamics, 2nd Int. JSME.Symp., Fluid Machinery and Fluidics, (Tokyo) (1972) 15–23.
4. H. Hasimoto, A sphere theorem on the Stokes equation for axisymmetric viscous flow, *J. Phy. Soc. Jpn.* 11 (1956) 793–797.
5. W. D. Collins, Note on a sphere theorem for the axisymmetric Stokes flow of a viscous fluid. *Mathematika* 5 (1958) 118–121.
6. D. Palaniappan, S. D. Nigam and T. Amaranath and R. Usha, Sphere Theorem for Stokes flow. *Mech. Res. Comm.* 17 (1990) 173–174
7. D. Palaniappan, S. D. Nigam and T. Amaranath and R. Usha, Lamb's solution of Stokes's equation: A sphere theorem. *Q. J. Mech. Appl. Math.* 45 (1992) 47–56.
8. A.T. Chwang, On spherical inversions of polyharmonic functions. *Q. Appl. Math.* 44 (1987) 793–799.
9. R. Usha and S. D. Nigam, Flow in a spherical cavity due to a stokeslet. *Fluid Dyn. Res.* 11 (1993) 75–78.
10. B. S. Padmavathi, T. Amaranath and D. Palaniappan, Motion inside a liquid sphere: internal singularities. *Fluid Dyn. Res.* 15 (1995) 167–176.
11. H. Hasimoto, Sphere theorem on the Stokes equation for three-dimensional viscous flow. *J. Phys. Soc. Jpn.* 61 (1992) 3027–3029.
12. H. Hasimoto, Addendum to Sphere theorem on the Stokes equation for three-dimensional viscous flow. *J. Phys. Soc. Jpn.* 64 (1995) 3583–3584.
13. H. Lamb *Hydrodynamics*, (6th edition) New York: Dover (1945) 738pp.